Pergamon

Topology Vol. 37, No. 1, pp. 63–73, 1998

0040-9383/97 \$19.00 + 0.00

PII: S0040-9383(97)00006-2

UPPER BOUNDS IN THE THEORY OF UNKNOTTING OPERATIONS

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(*Received* 8 *February* 1996)

1. INTRODUCTION

In the last decade, there has been some progress in the theory of (generalised) crossing changes on knots and links in S^3. For example, in [11], Scharlemann proved that a composite knot cannot be unknotted by a single crossing change. Results have also been discovered for satellite knots [13], tangle composite knots [3, 6], totally knotted knots [14] and fibred knots [7]. A very general theorem was proved in [7] which unified and generalised many of these results. However, there seem to be very few theorems which provide unknotting information about *every* knot and link in S^3. It is the purpose of this paper to fill that gap. We shall be considering the following generalisation of a crossing change.

Definition. Let K be an oriented knot or link in S^3. Let D be a disc embedded in S^3. Suppose that the boundary $L = \partial D$ is disjoint from K. If $q \in \mathbb{N}$, then $1/q$ surgery on L is termed a *twist about L of order q*. If K intersects D at two points of opposite sign, then L is termed a *crossing link*. A twist of order q about a crossing link is shown in Fig. 1. A twist of order one about a crossing link is the standard notion of a crossing change.

We shall address the following problem. For a given oriented link K and natural number q, is it possible to simplify K by applying a twist of order q about some crossing link L? The obvious simplification is where the link K' which results from the twist is actually the unknot (hence the term "unknotting operation"). However, we consider two further types of simplification. The first is where K is a non-split link, but K' is a split link. In this case, we have decomposed the link into "smaller" pieces. The second type of simplification arises by considering the Euler characteristic $\chi(K)$ of an oriented link K. This is defined to be the maximal Euler characteristic of any Seifert surface for K. (What precisely we mean by a Seifert surface in this context is given below.) If $\chi(K') > \chi(K)$, then this too is viewed as a simplification. In either of these two cases, we find, in Theorem 2.4, an upper bound on q which depends only on K, not on L. We therefore obtain "unknotting" information about K. We compute this bound in a number of cases and demonstrate that, in these cases, it is sharp. When applied to specific classes of knots, such as fibred knots and totally knotted knots, this result generalises a number of known results from [7, 14].

The bound on q is more than just a theoretical upper bound; it can be very easily calculated. For example, if we are given a single alternating diagram of K, then an upper bound on q can be read directly from the diagram. This is what is investigated in Section 3. Again, a specific example is examined, and the inequality is shown to be sharp here. This result should be compared with the main theorem of [8]. There, the generalised crossing

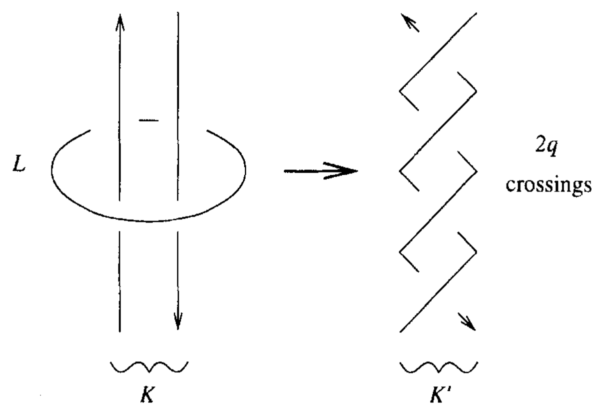


Fig. 1.

changes which were considered were shown to be related to the Jones polynomial of a link. In particular, an upper bound on q could be deduced for alternating knots which depended only on their crossing numbers.

Throughout this paper, we shall assume that $K \cup L$ is a non-split link. This is the only situation that interests us. For, if a two-sphere separates L from K, then a twist about L does not change the link. If a two-sphere in $S^3 - (K \cup L)$ separates two components of K, then a twist about L always yields a split link. Thus, it is natural to consider the case where $K \cup L$ is not split.

2. THE MAIN THEOREM

Throughout this paper, links are closed locally-flat 1-submanifolds of S^3, and a knot is a link with one component. We shall insist that a Seifert surface of an oriented link K is oriented in such a way that its boundary receives the same orientation as that of K. A Seifert surface need not be connected, but it has no closed components. An understanding of the theory of sutured manifolds as developed in [12, Sections 0–7] will be assumed.

Before we can state the main theorem of this paper, we must first establish some terminology.

Definition 2.1. Let M be a compact irreducible 3-manifold with boundary. A finite set $\mathscr{D}$ of disjoint compression discs for ∂M is said to be *adequate* if $M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D}))$ is boundary-irreducible. We say that $\mathscr{D} = \emptyset$ is adequate if ∂M is incompressible.

Example 2.2. Let M be a handlebody of genus g, and let $\mathscr{D}$ be a set of g disjoint compression discs for ∂M such that $M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D}))$ is a ball. Then $\mathscr{D}$ is obviously adequate.

It is well known that every compact irreducible 3-manifold with boundary has an adequate set of compression discs. To construct such a set, simply continue to boundary-reduce the manifold, until it is boundary-irreducible. This process eventually terminates. For we may assume that each compression disc is disjoint from previous ones, and so the collection $\mathscr{D}$ is a set of disjoint discs properly embedded in M. It is clear that for each disc D in $\mathscr{D}$, ∂D is essential in ∂M and that no two such curves are parallel in ∂M. There is an upper bound on the number of disjoint non-parallel essential simple closed curves in the surface ∂M, and so $\mathscr{D}$ is a finite adequate set.

Definition 2.3. Let M be a compact irreducible 3-manifold with boundary. Suppose that γ is a set of disjoint simple closed curves in ∂M. Let $\mathscr{D}$ be an adequate set of compression discs for ∂M. Let the integer $c(\mathscr{D},\gamma)$ be

$$c(\mathscr{D},\gamma) = \max\{\,|\partial D \cap \gamma|: D \in \mathscr{D}\}.$$

If $\mathscr{D} = \emptyset$, then we take $c(\mathscr{D},\gamma)$ to be zero. Let the integer $c(M,\gamma)$ be given by

$$c(M,\gamma) = \min\{c(\mathscr{D},\gamma)\text{: } \mathscr{D} \text{ is an adequate set of compression discs for } M\}.$$

For a given oriented link K in S^3, let $\mathscr{S}$ be the set of maximal Euler characteristic Seifert surfaces, where two such surfaces are identified if there is a homeomorphism of S^3 which takes one to the other. It was proved in [15] that a simple knot K has, up to ambient isotopy keeping K fixed, a finite set of minimal genus Seifert surfaces, and therefore, in this case, $\mathscr{S}$ is finite. For each S belonging to $\mathscr{S}$, let $M_S = S^3 - \operatorname{int}(\mathscr{N}(S))$. Then, ∂M_S is homeomorphic to two copies of S glued along ∂S. Let γ in ∂M_S be the copy of ∂S. We are now in a position to state the main theorem of this paper.

THEOREM 2.4. *Let K be an oriented link in S^3. Let K' be a link obtained from K by a twist of order q about some crossing link L, where $K \cup L$ is not a split link. If $\chi(K') > \chi(K)$ or K' is a split link, then we have the following inequality*:

$$q \leqslant \sup_{S \in \mathscr{S}} \frac{c(M_S,\gamma)}{2}.$$

Furthermore, suppose that S is an arbitrary maximal Euler characteristic Seifert surface for K, and that K' is either the unknot or split. Then

$$q \leqslant \max\left\{\frac{-\chi(S)+2}{2}, \frac{c(M_S,\gamma)}{2}\right\}.$$

Note that when $\mathscr{S}$ is finite, the first inequality gives an upper bound on q. In all cases, the second inequality gives such a bound. Note also that the bounds depend only on K and not on L.

The first inequality can be seen as a generalisation of two other theorems. In [14], a knot was said to be *totally knotted* if for every minimal genus Seifert surface S, M_S has incompressible boundary. In this case, the only adequate set of compression discs for M_S is the empty set, and so $c(M_S,\gamma)$ is zero. Thus, Theorem 2.4 has the following corollary, which was first proved in [14].

THEOREM (Scharlemann and Thompson [14, 3.4]). *No crossing change can lower the genus of a totally knotted knot.*

If K is a fibred knot, there is only one minimal genus Seifert surface S up to homeomorphism of S^3 (see [1, Theorem 5.1]). In this case, M_S is homeomorphic to the handlebody $S \times I$, and this homeomorphism takes γ to $\partial S \times \{0\}$. There exists a set $\mathscr{D}$ of disjoint compression discs for $\partial(S \times I)$, such that $c(\mathscr{D},\gamma) = 2$ and such that $S \times I - \operatorname{int}(\mathscr{N}(\bigcup \mathscr{D}))$ is a ball. Such a collection of discs is adequate. Thus, in Theorem 2.4, we must have $q \leqslant 1$, and so we have the following result, a more general version of which appeared in [7].

THEOREM (Lackenby [7, 5.2]). *Let K be a non-trivial fibred knot in S^3. Then, a twist of order greater than one about a crossing link cannot reduce the genus of K. In particular, it cannot unknot K.*

We now embark upon some preliminaries for the proof of Theorem 2.4. These are elementary but important.

LEMMA 2.5. *Let M be a compact irreducible 3-manifold with boundary, and let $\mathscr{D}$ be an adequate set of compression discs for ∂M. Then, given any compression disc E for ∂M, there exists a sequence of boundary compressions on E and ambient isotopies so that, after these operations, E has become a union of parallel copies of discs in $\mathscr{D}$.*

Proof. Instead of just considering a single compression disc E, it is more convenient to consider a collection of such discs. Therefore, let $\mathscr{E}$ be a finite non-empty set of disjoint compression discs for ∂M. Ambient isotope the discs of $\mathscr{E}$ so that they are in general position with respect to $\mathscr{D}$. Then $\bigcup \mathscr{E} \cap \bigcup \mathscr{D}$ is a set of curves. We shall prove the claim by induction on $N = |\bigcup \mathscr{E} \cap \bigcup \mathscr{D}|$.

If $N = 0$, then the discs of $\mathscr{E}$ lie in $M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D}))$, which has incompressible boundary. Hence, for each disc E of $\mathscr{E}$, ∂E bounds a disc D_E, say, in $\partial(M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D})))$. Since M is irreducible, $D_E \cup E$ bounds ball in $M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D}))$. Now, $D_E \cap \mathscr{N}(\bigcup \mathscr{D})$ is non-empty, since E is a compression disc for M. If $D_E \cap \mathscr{N}(\bigcup \mathscr{D})$ is a single disc, then E is a parallel to a disc of $\mathscr{D}$. Amongst the remaining discs E such that $D_E \cap \mathscr{N}(\bigcup \mathscr{D})$ is more than one disc, pick one so that D_E is innermost in $\partial(M - \mathrm{int}(\mathscr{N}(\bigcup \mathscr{D})))$. We may boundary compress this disc away from the rest of $\bigcup \mathscr{E}$ so that it becomes a union of discs each parallel to a disc of $\mathscr{D}$. Repeat this procedure as many times as required. This proves the lemma when $N = 0$.

Thus, we may assume that N is non-zero. Suppose that $\bigcup \mathscr{E} \cap \bigcup \mathscr{D}$ contains a simple closed curve. Then, pick one innermost on some disc D of $\mathscr{D}$. This bounds a subdisc D' of D and a subdisc E' of some disc E of $\mathscr{E}$. Then, $E' \cap D'$ is a sphere bounding a ball, the interior of which is disjoint from $\bigcup \mathscr{E}$, using which we may reduce N. Therefore, we may assume that $\bigcup \mathscr{E} \cap \bigcup \mathscr{D}$ is a set of arcs. Pick one, α, extrememost in some disc D of $\mathscr{D}$. Then $D - \mathrm{int}(\mathscr{N}(\alpha))$ is a union of two discs, one of which, D', misses $\bigcup \mathscr{E}$. Suppose that α lies in the disc E of $\mathscr{E}$. Let E_1 and E_2 be the set of discs which result from boundary compressing E along D'. If either E_1 or E_2 is not a compression disc for M, then we may ambient isotope $\bigcup \mathscr{E}$ so that the arc α has been removed, and N has been reduced. Thus, we may assume that both E_1 and E_2 are compression discs for ∂M. Let $\mathscr{E}'$ be $\mathscr{E} \cup \{E_1, E_2\} - E$. Then $|\bigcup \mathscr{E}' \cap \bigcup \mathscr{D}| < N$. Thus, the lemma is proved by induction. □

LEMMA 2.6. *Let F be a compact (possibly disconnected) surface properly embedded in a compact 3-manifold M. Suppose that F boundary compresses to F'. Let L be a knot in M. If there exists an ambient isotopy of L in M which pulls it clear of F', then there exists an ambient isotopy of L in M which pulls it clear of F.*

Proof. Let $\partial M \times I$ be a collar on ∂M. We may assume that F' respects this product structure, that is $F' \cap (\partial M \times I) = \partial F' \times I$. We may assume that F and F' differ only within $\partial M \times [0, \frac{1}{2}]$; see Fig. 2. Perform the ambient isotopy of L in M so that it is disjoint from F'.

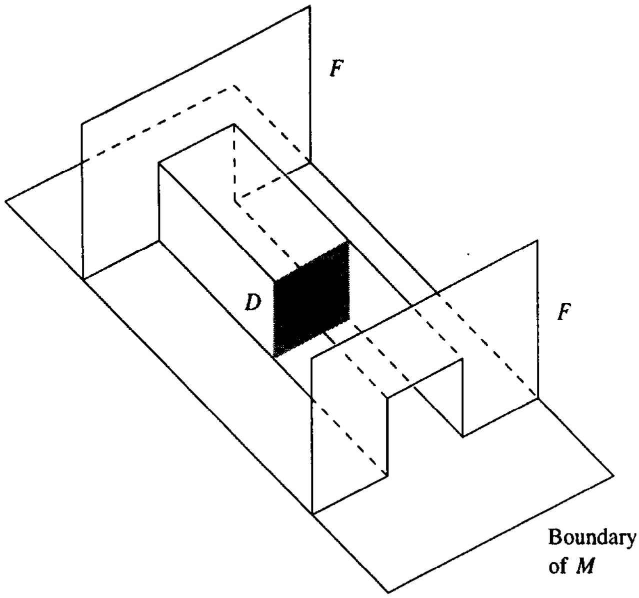


Fig. 2.

Follow this by another isotopy supported in the collar on ∂M so that, after the isotopy, $L \cap (\partial M \times [0, \frac{1}{2}]) = \emptyset$. This does not introduce any new points of $L \cap F'$. Thus, after this isotopy, L is disjoint from F. $\square$

The following immediate corollary of Lemmas 2.5 and 2.6 is the reason for considering adequate sets of compression discs. It, together with Theorem 1.4 of [7], is the key ingredient in the proof of Theorem 2.4.

Corollary 2.7. *Let $\mathscr{D}$ be a finite adequate set of compression discs for a compact irreducible* 3*-manifold M. Let E be an arbitrary compression disc for ∂M. Suppose that there is an ambient isotopy of a knot L in M, so that after the isotopy, L is disjoint from $\bigcup \mathscr{D}$. Then, there is an ambient isotopy of L in M which pulls L clear of E.*

We shall also be needing the following result. This is a version of a theorem of Scharlemann and Thompson [14, Proposition 3.1]. It is not explicitly stated in [14], but their argument readily implies it.

Proposition 2.8. *Let K be an oriented link in S^3. Let K' be a link obtained from K by a twist of order q about some crossing link L, such that $K \cup L$ is not a split link. If $\chi(K') > \chi(K)$, then there is a maximal Euler characteristic Seifert surface S for K which appears as in Fig.* 3 *near L.*

Proof of Theorem 2.4. Suppose that $\chi(K') > \chi(K)$ or that K' is split. If $\chi(K') > \chi(K)$, then Proposition 2.8 implies that there exists a maximal Euler characteristic Seifert surface S for K, disjoint from L, with the following properties.

1. M_S has a compression disc E.

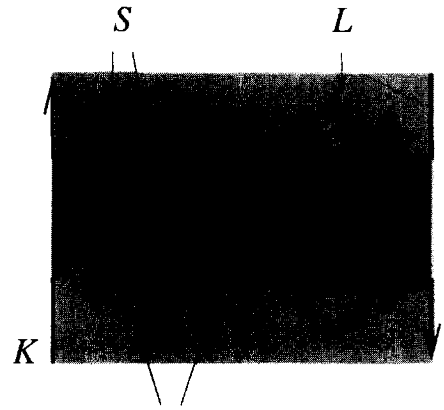


Fig. 3.

2. There is no ambient isotopy of L in M_S such that, after the isotopy, L is disjoint from E.
3. L bounds a disc D in S^3 which intersects S in a single arc.

We shall show that, when K' is split, there is also a maximal Euler characteristic Seifert surface S for K, disjoint from L, with the same three properties. Pick a Seifert surface for K with maximal Euler characteristic amongst those that miss L. We may find such a surface S so that (3) holds. Then $D - \mathrm{int}(\mathcal{N}(S))$ is an annulus, one boundary component of which is L, the other of which lies in ∂M_S. On twisting about L, S becomes a Seifert surface S' for K'. Let γ' be the curves in $M_{S'}$ corresponding to $\partial S'$. Let $\mathcal{N}(L')$ be the surgery solid torus in $M_{S'}$, and let L' be its core. Note that there is a homeomorphism of M_S onto $M_{S'}$ which maps L onto L', but does not map γ onto γ'. Now, the argument of [5] and Property (3) imply that $S^3 - \mathrm{int}(\mathcal{N}(K))$ is S_L-atoroidal [4]. Corollary 2.4 of [4] implies that K is not split and that S is a maximal Euler characteristic Seifert surface for K. Therefore, M_S is irreducible, since S has no closed components. Hence, $M_{S'}$ is irreducible. Since K' is split, a simple innermost curve argument implies that S' must compress. Hence, there is a compression disc E' for $M_{S'}$ which is disjoint from γ'. Let E be the copy of E' in M_S. Then E is a compression disc for M_S, which establishes (1). Note that there cannot be an ambient isotopy of L in M_S such that, after the isotopy, L is disjoint from E. For then, we could construct an ambient isotopy of E' in $M_{S'}$ such that after the isotopy E' is disjoint from $\mathcal{N}(L')$. We may assume that such an isotopy is fixed on $\partial M_{S'}$. Hence, the resulting disc is also disjoint from γ'. But then, S' has a compression disc disjoint from L' and so S compresses, which contradicts the fact that it is a maximal Euler characteristic Seifert surface for K. This establishes (2).

Let $M_1 = S^3 - \mathrm{int}(\mathcal{N}(L \cup K))$. Then M_1 can be considered as a sutured manifold with $R_- = \partial M_1$, and $R_+ = \emptyset$. It is taut because $L \cup K$ is not a split link. Construct the taut decomposition

$$(M_1, \emptyset) \xrightarrow{S} (M_S - \mathrm{int}(\mathcal{N}(L)), \gamma).$$

Since $S^3 - \mathrm{int}(\mathcal{N}(K))$ is S_L-atoroidal, any torus in $M_S - \mathrm{int}(\mathcal{N}(L))$ which is I-cobordant to $\partial\mathcal{N}(L)$ is parallel to $\partial\mathcal{N}(L)$. Note that $(M_{S'}, \gamma')$ is not taut, since there is a compression disc for $\partial M_{S'}$ disjoint from γ'. Let $\mathcal{D}$ be an adequate set of compression discs for M_S.

CLAIM. $q \leqslant c(\mathscr{D}, \gamma)/2$.

Case 1: *$H_2(M_S - \mathrm{int}(\mathscr{N}(L)), \partial M_S)$ is trivial.* Then, it is argued in Theorem 5.1 of [12] that M_S is a solid torus of which L is a core. Further, γ is a non-empty collection of essential curves in ∂M_S, each of which is parallel in $M_S - \mathrm{int}(\mathscr{N}(L))$ to $1/q$ in $\partial\mathscr{N}(L)$. There are an even number of such sutures. Each disc D of $\mathscr{D}$ touches γ at least $q|\gamma|$ times, and so $c(\mathscr{D}, \gamma) \geqslant q|\gamma|$. Thus,

$$q \leqslant \frac{c(\mathscr{D}, \gamma)}{|\gamma|} \leqslant \frac{c(\mathscr{D}, \gamma)}{2}.$$

This proves the claim in this case.

Case 2: *$H_2(M_S - \mathrm{int}(\mathscr{N}(L)), \partial M_S)$ is non-trivial.* We can then apply Theorem 1.4(a) of [7] to deduce that there exists an ambient isotopy of L in M_S so that, for each disc D in $\mathscr{D}$, the following inequality holds after the isotopy:

$$|L \cap D| \leqslant \frac{-2 + |D \cap \gamma|}{2(q-1)}.$$

By Corollary 2.7 and Property (2) above, L cannot be isotoped off some disc D of $\mathscr{D}$. For this disc, the left-hand side of the inequality is non-zero, and hence

$$2(q-1) \leqslant -2 + |D \cap \gamma|,$$

which implies that

$$q \leqslant \frac{|D \cap \gamma|}{2} \leqslant \frac{c(\mathscr{D}, \gamma)}{2}.$$

This proves the claim in this case.

The inequality of the claim holds for all adequate sets $\mathscr{D}$, and so

$$q \leqslant \frac{c(M_S, \gamma)}{2} \leqslant \sup_{S \in \mathscr{S}} \frac{c(M_S, \gamma)}{2}.$$

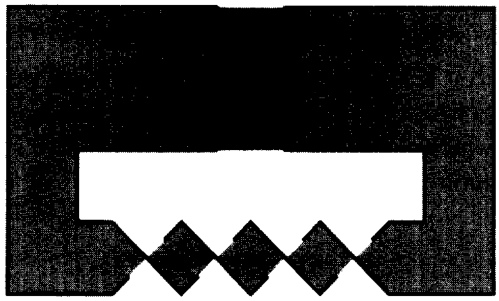
This proves the first inequality.

Now suppose that S is an arbitrary maximal Euler characteristic Seifert surface for K, and that K' is either the unknot or split. Then the argument of Corollary 3.6 of [7] implies that, if $q > (-\chi(S) + 2)/2$, then there is an ambient isotopy of L in $S^3 - \mathrm{int}(\mathscr{N}(K))$ such that, after the isotopy, L is disjoint from S. Properties (1)–(3) above are established exactly as in the case above where K' was split. More specifically, we ambient isotope L in $S^3 - \mathrm{int}(\mathscr{N}(K))$, so that (3) holds. Now, Corollary 2.4 of [4] implies that K is neither the unknot nor split. But K' is either the unknot or split. Hence, the Seifert surface S' which results from the twist must compress. The above analysis implies that

$$q \leqslant \frac{c(M_S, \gamma)}{2}.$$

and hence the second inequality holds. □

Example 2.9. Here, we consider an application of the first inequality of Theorem 2.4. Let K be the twist knot T_n [10, p. 112] with $|n|$ twists, where $n \neq 0$. It was proved in [9] that K has a minimal genus Seifert surface S which is unique up to an isotopy of S^3 which keeps K fixed; see Fig. 4.



2|n| crossings

Fig. 4.

Now, M_S is a handlebody of genus 2. There exist two disjoint compression discs, D_1 and D_2, for M_S such that $M_S - \mathrm{int}(\mathcal{N}(D_1 \cup D_2))$ is a ball. These form an adequate set. Also, $|D_1 \cap \gamma|$ and $|D_2 \cap \gamma|$ are equal to 2 and $2|n|$, respectively. Thus, $c(\{D_1, D_2\}, \gamma)$ is equal to $2|n|$. Therefore, $c(M_S, \gamma) \leqslant 2|n|$. Thus, Theorem 2.4 implies that, if a twist of order q about some crossing link unknots K, then $q \leqslant |n|$. This inequality is sharp, since there is such a twist of order $|n|$ which unknots K.

3. UPPER BOUNDS FROM LINK DIAGRAMS

In this section, we consider an application of the second inequality of Theorem 2.4. We deduce an upper bound on q which can be read very easily from a single link diagram of a certain type. This type includes all non-split alternating diagrams.

Let K be an oriented link in S^3 and let D be a diagram of K in S^2. Let N be a regular neighbourhood of the image of K in S^2. Then $S^2 - \mathrm{int}(N)$ is a set of regions $\mathcal{R}$. We wish to assign an integer $c(R)$ to each region R. The boundary of a region naturally has the structure of an oriented graph, with the vertices coming from the crossings and the edges coming from the rest of the image of K in S^2. We say that a vertex is of Type 1 if the two edges which it abuts are oriented inconsistently, and of Type 2 otherwise (see Fig. 5).

Define $c(R)$ to be the number of vertices of Type 1 which lie on the boundary of R.

THEOREM 3.1. *Let K be an oriented link in S^3 (other than the unknot) and let D be a non-split diagram for K in S^2, with regions $\mathcal{R}$. Suppose that the Seifert algorithm yields a maximal Euler characteristic Seifert surface S for K. Suppose that a twist of order q about some crossing link L yields the unknot or a split link. Then*

$$q \leqslant \max\left\{\frac{-\chi(S)+2}{2}, \max_{R \in \mathcal{R}} \frac{c(R)}{2}\right\}.$$

Proof. Note that K is not a split link, since any maximal Euler characteristic Seifert surface for a split link is disconnected, whereas S is connected. Hence, if $K \cup L$ is split, then twisting about L does not change K. We may therefore assume that $K \cup L$ is non-split.

We shall consider an explicit construction of $\mathcal{N}(S)$ as a subset of S^3. In the Seifert algorithm, the crossings of D are removed in a way consistent with the orientation on K. The result is a number of simple closed curves, the Seifert circles, in S^2. Then S is realised as

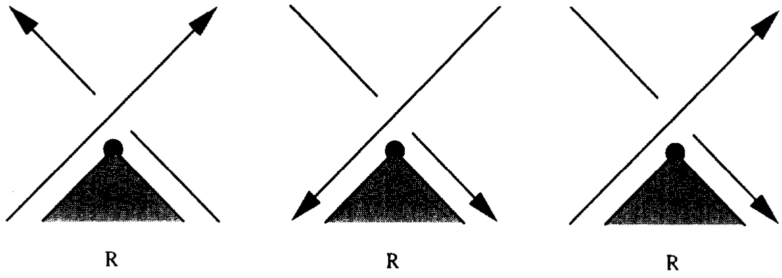


Fig. 5.

a union of several half-twisted bands, one for each crossing, together with a disc above S^2 for each Seifert circle. Thus, $\mathcal{N}(S)$ can be realised as a regular neighbourhood of these discs to which a number of solid "cylinders" are attached, one cylinder for each crossing. Now, each region R of the diagram is a disc which we may assume to be properly embedded in M_S. Shift each disc R by a small amount so that it lies above S^2. Note that the resulting discs $\{R\colon R \in \mathcal{R}\}$ are disjoint from one another, and that $|R \cap \gamma| = c(R)$. Now, $M_S - \operatorname{int}(\mathcal{N}(\bigcup\{R\colon R \in \mathcal{R}\}))$ is a collection of balls, and therefore M_S is a handlebody. So, some subset $\mathcal{D}$ of $\{R\colon R \in \mathcal{R}\}$ is an adequate set. The theorem now follows from the second inequality of Theorem 2.4. □

The right-hand side of the inequality in Theorem 3.1 can easily be read from the diagram D; see Example 3.4. Now, in the proof of Theorem 3.1, $M_S - \operatorname{int}(\mathcal{N}(\bigcup\{R\colon R \in \mathcal{R}\}))$ is a *collection* of balls. We deduce that $\mathcal{R}$ is actually needlessly large. There exist subsets $\mathcal{R}'$ of $\mathcal{R}$ which still form adequate sets. What are these subsets? Let $\mathcal{C}$ be the set of Seifert circles for the diagram D. Then, each component X of $S^2 - \bigcup\mathcal{C}$ contains a number of regions $\mathcal{R}_X$. We shall call a subset $\mathcal{R}'$ of $\mathcal{R}$ *sufficient* if, for each component X of $S^2 - \bigcup\mathcal{C}$, $\mathcal{R}_X - \mathcal{R}'$ is exactly one region. It is clear that $M_S - \operatorname{int}(\mathcal{N}(\bigcup\{R\colon R \in \mathcal{R}'\}))$ is a ball for any sufficient subset $\mathcal{R}'$ of $\mathcal{R}$. Thus, we have the following strengthening of Theorem 3.1.

THEOREM 3.2. *Let K be an oriented link in S^3 (other than the unknot) and let D be a non-split diagram for K in S^2, with regions $\mathcal{R}$. Let $\mathcal{R}'$ be a sufficient subset of $\mathcal{R}$. Suppose that the Seifert algorithm yields a maximal Euler characteristic Seifert surface S for K. Suppose that a twist of order q about some crossing link L yields the unknot or a split link. Then*

$$q \leqslant \max\left\{\frac{-\chi(S)+2}{2},\ \max_{R\in\mathcal{R}'}\frac{c(R)}{2}\right\}.$$

The following corollary is an application of Theorem 3.1 to non-split alternating diagrams. There is also a version which uses Theorem 3.2.

COROLLARY 3.3. *Let D be a non-split alternating diagram of an oriented link K in S^3 (other than the unknot). Let $\mathcal{R}$ be the set of its regions, let $n(D)$ be the number of crossings and let $s(D)$ be the number of Seifert circles. Suppose that a twist of order q about some crossing link L yields the unknot or a split link. Then*

$$q \leqslant \max\left\{\frac{n(D)-s(D)+2}{2},\ \max_{R\in\mathcal{R}}\frac{c(R)}{2}\right\}.$$

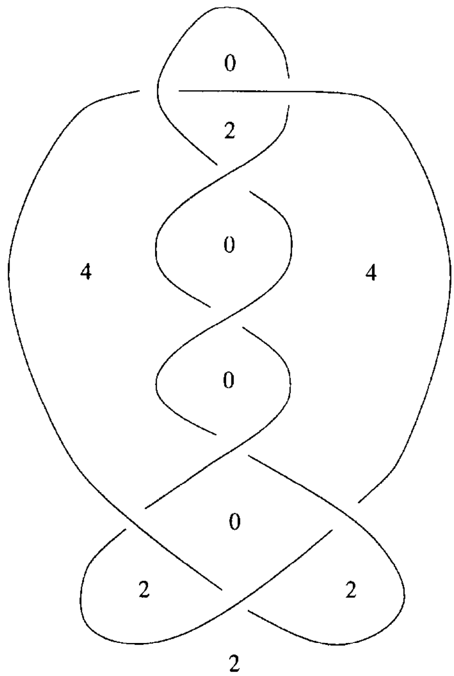


Fig. 6.

Proof. Let S be the Seifert surface constructed via the Seifert algorithm. Note that $\chi(S) = s(D) - n(D)$. To apply Theorem 3.1, we need to check that S has maximal Euler characteristic. It was proved in [2] that the Seifert surface S has minimal genus. The following equality is trivial for a compact orientable surface S':

$$\text{genus}(S') = \frac{-\chi(S') - |\partial S'|}{2} + |S'|.$$

Thus, the corollary is proved if we can show that K does not bound a disconnected Seifert surface, since then, for all Seifert surfaces for K, the final term is 1. However, if K does bound a disconnected Seifert surface, then it has zero Alexander polynomial, and it was shown in [2] that the Alexander polynomial of K is non-trivial. □

Example 3.4. An alternating diagram D of the knot 8_6 of [10] is shown in Fig. 6. The value $c(R)$ for each region R is shown in the figure. (Note that, for a knot, it is independent of orientation.) Also, $n(D) = 8$ and $s(D) = 5$. Corollary 3.3 implies therefore that it cannot be unknotted by a twist of order greater than 2 about any crossing link. This inequality is sharp, since there is a twist of order 2 about a crossing link which unknots 8_6.

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